REMARKS ON OSMOSIS, QUANTUM MECHANICS, AND GRAVITY

ROBERT CARROLL
UNIVERSITY OF ILLINOIS, URBANA, IL 61801

Abstract. Some relations of the quantum potential to Weyl geometry are indicated with applications to the Friedmann equations for a toy quantum cosmology. Osmotic velocity and pressure are briefly discussed in terms of quantum mechanics and superfluids with connections to gravity.

1. REMARKS ON WEYL GEOMETRY

We begin with some features of Weyl geometry and the Dirac-Weyl theory following [54, 55, 56, 84] (cf. also [1, 2, 3, 4, 8, 12, 15, 16, 17, 19, 20, 23, 24, 25, 29, 77, 81, 85, 86, 87, 95]). Note that the paper is in part an expansion of some ideas in [24] and we correct some notational confusion from [17, 24]. There is a curious issue here of too much or too little, which is compounded by various, sometimes conflicting, notations and we feel that it is well advised to simply follow the extensive development of Israelit-Rosen which will keep the notation under control and benefit from insights therein. The background involves looking at a metric tensor $g_{ab} = g_{ba}$ and a length connection vector w_μ. If a vector is displaced by dx^ν then

$$dB^\mu = -B^\sigma \hat{\Gamma}^\mu_{\sigma\nu} dx^\nu;\ dB = B w_\nu dx^\nu \tag{1.1}$$

One can also write (**1A**) $d(B^2) = 2B^2 w_\nu dx^\nu$ and this all requires that (**1B**) $g_{\mu\sigma}\hat{\Gamma}^\sigma_{\nu\sigma} + g_{\nu\sigma}\hat{\Gamma}^\sigma_{\mu\sigma} = \partial_\sigma g_{\mu\nu} - 2g_{\mu\nu}w_\sigma$. Then assuming that $\hat{\Gamma}^\sigma_{\mu\nu} = \Gamma^\sigma_{\mu\nu}$ there results

$$\hat{\Gamma}^\sigma_{\mu\nu} = \Gamma^\sigma_{\mu\nu} + g_{\mu\nu}w^\sigma - \delta^\sigma_\nu w_\mu - \delta^\sigma_\mu w_\nu \tag{1.2}$$

where $\Gamma^\sigma_{\mu\nu}$ denotes the standard Riemannian Christoffel symbol. The covariant derivative for g_{ab} is written ∇_α or e.g. $B^\mu_{;\nu} = \nabla_\nu B^\mu$ (and involves $\Gamma^\sigma_{\mu\nu}$) whereas with the Weyl connection (1.2) one forms a co-covariant derivative of a vector

$$B^\mu_{:\nu} = \partial_\nu B^\mu + B^\sigma \hat{\Gamma}^\mu_{\sigma\nu};\ B_{\mu,:\nu} = \partial_\nu B_\mu - B_\sigma \hat{\Gamma}^\sigma_{\mu\nu} \tag{1.3}$$

Date: April, 2011.
email: rcarroll@math.uiuc.edu.

so that

$$B^{\mu}_{:\nu} = B^{\mu}_{;\nu} + B^{\mu}(g_{\sigma\nu}w^{\mu} - \delta^{\mu}_{\nu}w_{\sigma} - \delta^{\mu}_{\sigma}w_{\nu}) \tag{1.4}$$

Note also

$$g_{\mu\nu:\sigma} = 2g_{\mu\nu}w_{\sigma};\ g^{\mu\nu}_{:\sigma} = -2g^{\mu\nu}w_{\sigma} \tag{1.5}$$

Now under Weyl gauge transformations one has a length change (**1C**) $B \to \hat{B} \sim \tilde{B} = e^{\lambda(x)}B$ with $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{2\lambda}g_{ab}$ and $g^{ab} \to e^{-2\lambda}g^{ab}$. Writing $e^{2\lambda(x)} = f(x)$ there is a calculation from [28] for lengths $\ell \to \hat{\ell} = f^2\ell$

$$2\hat{\ell}d\hat{\ell} \sim f_{\alpha}dx^{\alpha}\ell^2 + 2f\ell d\ell \Rightarrow \frac{d\hat{\ell}}{\hat{\ell}} \sim \frac{1}{2}\partial_{\alpha}log(f)dx^{\alpha} + w_{\alpha}dx^{\alpha} \Rightarrow \hat{w}_{\alpha} = \tag{1.6}$$
$$= w_{\alpha} + \frac{1}{2}\partial_{\alpha}log(f) = w_{\alpha} + \partial_{\alpha}\lambda(x)$$

REMARK 1.1. In Weyl geometry consider now a scalar S of Weyl weight n defined via $\tilde{S} = exp(n\lambda)S$ (e.g. $B = (g_{ab}B^aB^b)^{1/2} = \ell \to \hat{\ell} = e^{\lambda}\ell$) and $\partial_{\mu}\tilde{S} = exp(n\lambda)(\partial_{\mu}S + nS\partial_{\mu}\lambda)$. This is not covariant with respect to Weyl gauge transformations (WGT) unless $n = 0$ but one can define a gauge covariant derivative (**1D**) $S_{||\mu} = \partial_{\mu}S - nSw_{\mu}$ with $\tilde{S}_{||\mu} = exp(n\lambda)S_{||\mu}$ (using (1.6)). For a gauge covariant derivative of a vector S^{μ} of Weyl weight n, where $\tilde{S}^{\alpha} = exp(n\lambda)S^{\alpha}$, there results (**1E**) $S^{\mu}_{||\nu} = S^{\mu}_{:\mu} - nS^{\mu}w_{\nu} = exp(n\lambda)S^{\mu}_{||\nu}$ (note $\hat{\Gamma}^{\mu}_{\sigma\nu} - n\delta^{\mu}_{\sigma}w_{\nu}$ is actually a connection). ■

Dirac [28] (1973) introduced a β field of weight -1 (as a Lagrange type multiplier) in order to have a satisfactory scalar tensor action of the form (cf. [54], Chap. 4)

$$I = \int \sqrt{-g}d^4x[-\beta^2R + \sigma\beta^2w^cw_c + (\sigma+6))\partial_c\beta\partial^c\beta + 2\sigma\beta\partial_c\beta w^c + 2\Lambda\beta^4] \tag{1.7}$$

Note here that $\sqrt{-g}$ is of weight 4 and R of weight -2; electromagnetic terms have been omitted (also $\hbar = c = 1$). It is this factor β which was identified as a quantum mass $\mathfrak{M} = mexp(\mathfrak{Q})$ in [87] (and subsequently utilized in [15, 16, 17, 19, 20, 23, 24]). It's quantum significance was also noticed in [8] (cf. also [81]) in connection with conformal GR (GR stands for general relativity). Unfortunately a different notation was used in [8] and in [81] (0009169), namely (**1F**) $\hat{g}_{ab} = \Omega^2(x)g_{ab}$ with $\Omega^2 = exp(-\psi)$ in [8] and $\Omega^2 = exp(\psi)$ in [81] (0009109) - a fact which I occasionally overlooked in subsequent work (cf. [24] for some errata). The latter paper [81] (0009169) contains some important material relating various actions in Einstein frame (EF) and Jordan frame (BD type equations). It was shown that an action

$$S_4 = \int d^4x\sqrt{-\hat{g}}e^{-\psi}\left[\hat{R} - \left(\alpha - \frac{3}{2}\right)|\hat{\nabla}\psi|^2 + 16\pi e^{-\psi}L_M\right] \tag{1.8}$$

was the only one (of 4 considered) which was invariant under transformations of units (length, time, and mass) and this was then dubbed a string frame action (under the assumption that strings are ultimate and inevitable - cf. [44]). Note that $\Omega^2 = exp(\psi) = \phi$ implies

$$\hat{g}_{ab;c} = \partial_c\psi\hat{g}_{ab} \Rightarrow w_c = \frac{1}{2}\partial_c\psi \tag{1.9}$$

In particular conformal Riemannian geometry as in S_4 is a WIST (Weyl integrable space time) as noted in [81] (0009169) and [73, 74].

Now S_4 arises in our work via a conformal map $g_{ab} \to \hat{g}_{ab} = \Omega^2 g_{ab}$ from an EF action (cf. [68]) (**1G**) $S = \int\sqrt{-g}d^4x[R - \alpha|\nabla\psi|^2 + 16\pi L_M]$ in the form (**1H**) $\hat{S} = \int\sqrt{-\hat{g}}d^4x[\hat{\phi}\hat{R} - (\omega/\hat{\phi})|\hat{\nabla}\hat{\phi}|^2 + 16\pi\hat{\phi}^2 L_M]$ where we will write out the theory here in terms of $\Omega^2 = \exp(\psi)$ following [19] (cf. also [15, 16] and note a few sign adjustments are needed in [17]). Recall first that $\phi \sim \beta^2/m^2 = \mathfrak{M}^2/m^2$ and $\hat{\phi} = \phi^{-1} = exp(-\psi)$ and note that $\sqrt{-g} = \hat{\phi}^2\sqrt{-\hat{g}}$ (cf. [15, 16]) and

$$\sqrt{-\hat{g}}\hat{\phi}\hat{R} = \hat{\phi}^{-1}\sqrt{-\hat{g}}\hat{\phi}^2\hat{R} = \hat{\phi}^{-1}\sqrt{-g}\hat{R} = \frac{\beta^2}{m^2}\sqrt{-g}\hat{R} \tag{1.10}$$

But via [28, 54] we can write (cf. also [17] for the pattern here)

$$\beta^2\hat{R} = \beta^2 R - 6\beta^2\nabla_\kappa w^\kappa + 6\beta^2 w^\kappa w_\kappa \tag{1.11}$$

and via$-\beta^2\nabla_{gk}w^\kappa = -\nabla_\kappa(\beta^2 w^\kappa) + 2\beta\partial_\kappa\beta w^\kappa$, with a vanishing divergence term upon integration, the first integral in $\hat{S}$ becomes

$$I_1 = \int\sqrt{-g}d^4x\left[\frac{\beta^2}{m^2}R + 12\beta\partial_\kappa\beta w^\kappa + 6\beta^2 w^\kappa w_\kappa\right] \tag{1.12}$$

The second integral is $(\gamma = \alpha - (3/2))$

$$I_2 = -\gamma\int\sqrt{-\hat{g}}d^4x\hat{\phi}\frac{|\hat{\nabla}\hat{\phi}|^2}{|\hat{\phi}|^2} = -\frac{4\gamma}{m^2}\int\sqrt{-g}d^4x|\hat{\nabla}\beta|^2 \tag{1.13}$$

since $\hat{\phi}_c = -\psi_c e^{-\psi}$, $\hat{\nabla}\psi = -\hat{\nabla}\hat{\phi}/\hat{\phi}$, $\hat{\phi} = \phi^{-1} = m^2\beta^{-2}$, and $\hat{\phi}/\hat{\phi} = -2\hat{\nabla}\beta/\beta$ (along with (1.10)). Via (**1D**) $\hat{\nabla}_c\beta = \partial_c\beta + w_c\beta$ for $\Pi(\beta) = -1$ (Π denotes the Weyl weight) and from $\beta^2 = m^2 exp(\psi)$ one has $\psi_c exp(\psi) = (\beta^2/m^2)\psi_c \Rightarrow \psi_c = 2\beta_c/\beta$. Hence via (1.9) $w_c = \beta_c/\beta$ (correcting [17]) and consequently $\hat{\nabla}_c\beta = 2\beta_c$. In any event

$$\hat{S}_{GR} = \frac{1}{m^2}\int\sqrt{-g}d^4x[\beta^2 R + 12\beta\partial_c\beta w^c + 6\beta^2 w^c w_c - 4\gamma|\hat{\nabla}\beta|^2 + 16\pi m^2 L_M] \tag{1.14}$$

(as in [17], p. 59 (6.6) and p. 237 (3.32)).

REMARK 1.2. There was some sign confusion in [17] which we have adjusted above. Thus equations (6.7)-(6.8) on p. 59 and (3.33))-(3.34) on

p. 237 should be altered as indicated below. ■

First we write $|\hat{\nabla}\beta|^2 = \beta^c\beta_c$ and from (1.9) and (1.14)

$$(1.15)\quad I_1 = \frac{1}{m^2}\int \sqrt{-g}d^4x[\beta^2 R + 18\partial_c\beta\partial^c\beta];\ I_2 = -16\gamma\int\sqrt{-g}d^4x\beta^c\beta_c$$

Consequently (recall $\gamma = \alpha - (3/2)$)

$$(1.16)\qquad \hat{S} = \frac{1}{m^2}\int\sqrt{-g}d^4x[\beta^2 R + 16\pi m^2 L_M + 2(21-8\alpha)\beta^c\beta_c]$$

Then we rewrite (1.7) in the form

$$(1.17)\qquad I = \int\sqrt{-g}d^4x[-\beta^2 R + 3(\sigma+2)\beta_c\beta^c + 2\Lambda\beta^4]$$

Ignoring L_M and Λ for the moment we see that $I = m^2\hat{S}$ provided that (**1I**) $3\sigma = 8(3-\alpha)$ and, a priori, α or σ can be arbitrary (modulo e.g. possible cosmological restrictions on α).

REMARK 1.3 The Brans-Dicke (BD) type theory of S_4 (written also as $\hat{S}_4$) was used for the Friedman equations in [24] for example where a relation $\sigma \sim -4\alpha$ was used (based on some sign confusion). Hence one should convert the Weyl-Dirac action (1.1) of [24] to (1.7) and use the (σ,α) relation (**1I**) in order to obtain the correct Friedmann equations for the toy quantum cosmology considered there (cf. Section 2). ■

2. REMARKS ON COSMOLOGY

In [24] the Friedmann equations were based on an S_4 type action

$$(2.1)\qquad \tilde{S} = \frac{1}{16\pi}\int\sqrt{-g}d^4x\left[R\Phi - \omega\frac{|\nabla\Phi|^2}{\Phi} + L_M\right]$$

where (**2A**) $L_M = -V(\Phi) + 16\pi\mathfrak{L}$ with Λ assumed to be suitably inserted in $\mathfrak{L}_M$ (cf. [33, 38]). An FRW metric ($c=1$)

$$(2.2)\qquad ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Sigma^2\right]$$

is used and one has field equations as in [24]

$$(2.3)\qquad G_{ab} = \frac{8\pi}{\Phi}T^M_{ab} + \frac{\omega}{\Phi^2}\left[\nabla_a\Phi\nabla_b\Phi - \frac{1}{2}g_{ab}\nabla^c\Phi\nabla_c\Phi + \right]$$

$$+\frac{1}{\Phi}(\nabla_a\nabla_b\phi - g_{ab}\Box\Phi) - \frac{V}{2\Phi}g_{ab}$$

where (**2B**) $T^M_{ab} = -(2/\sqrt{-g})(\delta/\delta g^{ab})(\sqrt{-g}\mathfrak{L}_M)$ (we ignore a possible shift $\sqrt{-g} \to \sqrt{g}$, etc. due to the signature in (2.2)). Variation of the action with respect to Φ gives

$$\frac{2\omega}{\Phi}\Box\Phi + R - \frac{\omega}{\Phi^2}\nabla^c\Phi\nabla_c\Phi - \frac{dV}{d\Phi} = 0 \tag{2.4}$$

with trace

$$R = -\frac{8\pi T_M}{\Phi} + \frac{\omega}{\Phi^2}\nabla^c\Phi\nabla_c\Phi + \frac{3\Box\Phi}{\Phi} + \frac{2V}{\Phi} \tag{2.5}$$

and using (2.5) to eliminate R from (2.4) yields

$$\Box\Phi = \frac{1}{2\omega+3}\left[8\pi T^M + \Phi\frac{dV}{d\Phi} - 2V\right] \tag{2.6}$$

Then using (2.2) one obtains (**2C**) $\nabla^c\Phi\nabla_c\Phi = -(\dot{\Phi})^2$ and $\Box\Phi = -(\ddot{\Phi} + 3H\dot{\Phi}) = -(1/a^3)(d/dt)(a^3\dot{\Phi})$. Assume now that (**2D**) $T^M_{ab} = (P^M + \rho^M)u_a u_b + P^M g_{ab}$ and then the time dependent component of the BD field equations gives a constraint equation

$$H^2 = \frac{8\pi}{3\Phi} + \frac{\omega}{6}\left(\frac{\dot{\Phi}}{\Phi}\right)^2 - H\frac{\dot{\Phi}}{\Phi} - \frac{k}{a^2} + \frac{V}{6\Phi} \tag{2.7}$$

Then, using $R = 6[\dot{H} + 2H^2 + (k/a^2)]$, there results

$$\dot{H} + 2H^2 + \frac{k}{a^2} = -\frac{4\pi T^M}{3\Phi} - \frac{\omega}{6}\left(\frac{\dot{\Phi}}{\Phi}\right)^2 + \frac{1}{2}\frac{\Box\Phi}{\Phi} + \frac{V}{3\Phi} \tag{2.8}$$

From (2.6), (2.7), and the trace equation (**2E**) $T^M = 3P^M - \rho^M$ one has then

$$\dot{H} = \frac{-8\pi}{(2\omega+3)\Phi}\left[(\omega+2)\rho^M + \omega P^M\right] - \frac{\omega}{2}\left(\frac{\dot{\Phi}}{\Phi}\right)^2 + \tag{2.9}$$
$$+2H\frac{\dot{\Phi}}{\Phi} + \frac{k}{a^2} + \frac{2}{2(2\omega+3)\Phi}\left(\Phi\frac{dV}{d\Phi} - 2V\right)$$

and (2.6) reduces to

$$\ddot{\Phi} + 3H\dot{\Phi} = \frac{1}{2\omega+3}\left[8\pi(\rho^M - 3P^M) - \Phi\frac{dV}{d\Phi} + 2V\right] \tag{2.10}$$

In order to apply this to our model (1.7) we look at these Friedmann equations for $\Phi = \hat{\phi}$ (with $\mathfrak{Q} \sim \psi$) where $(\beta^2/m^2) = \hat{\phi}^{-1} = exp(\psi) = exp(\mathfrak{Q})$. Then $\hat{\phi} = exp(-\mathfrak{Q}) = \Phi$ and equations (2.3)-(2.10) can be written in terms of the quantum potential $\mathfrak{Q}$ from (**2F**) $\mathfrak{Q} = (\hbar^2/m^2)(\Box|\Psi|/|\Psi|)$ with Ψ arising in the quantum process underlying (1.2) (cf. [2, 3, 4, 15, 16,

17, 20, 23, 24, 25, 85, 87, 95] and see also [6, 32]). We assume first that $\mathfrak{L}_M = 0$ and then one can write (**2G**) $\dot{\Phi} = -\dot{\mathfrak{Q}}\Phi$, $\ddot{\Phi} = (\dot{\mathfrak{Q}}^2 - \ddot{\mathfrak{Q}})\Phi$ and thence e.g.

$$(\dot{\mathfrak{Q}}^2 - \ddot{\mathfrak{Q}})\Phi - 3H\dot{\mathfrak{Q}}\Phi = \frac{1}{2\omega+3}\left[8\pi(\rho^M - 3P^M) - \Phi\frac{dV}{d\Phi} + 2V\right] \tag{2.11}$$

Note that (**2H**) $\omega = \alpha - (3/2)$ and $3\sigma = 8(3-\alpha) \Rightarrow \sigma = 12 - (8/3)\omega$. We see that the dynamics of Φ is determined in part by $V(dV/d\Phi) - 2V$ (which vanishes for $V = c\Phi^2 = cexp(-2\mathfrak{Q})$) and by ρ^M, P^M. Thus, in particular, it is possible to envision some cosmological behavior provided by a quantum background as in (1.7). We note also that Mannheim refers to intrinsically quantum mechanical gravity associated to general Weyl geometry with the Weyl tensor, etc. (cf. [5, 69]). We especially recommend Padmanabhan's paper [76] (1012.4476) and references there for a discussion of quantum mechanics and spacetime. Even more striking is the discussion in Grössing's book [46] (p. 123) which indicates (assuming an aether) how Einsteinian gravity can be considered a pure quantum phenomenon.

REMARK 2.1. It is interesting to note that via $\hat{\phi} = exp(-\mathfrak{Q})$ one has $\mathfrak{Q} \sim \psi$ and hence by (1.9) the Weyl vector is $w_c = (1/2)\partial_c\mathfrak{Q}$ inplying a geometric role for $\mathfrak{Q}$. In particular this seems to show a direct connection of quantum mechanics with gravity. Of course this is not surprising since the geometry of (1.7) was generated via quantum mechanics but it suggests that Weyl geometry itself may have an implicit quantum connection via $w_a = (1/2)\partial_a\mathfrak{Q}$ with $\mathfrak{Q}$ a quantum potential. This seems to be a new observation (although connections of QM to the Weyl- Ricci curvature arise (cf. [2, 3, 15, 16, 20, 25, 85, 95]). ∎

REMARK 2.2. In [87] there are also joint field equations involving the conformal factor Ω^2 (expressed via ϕ or ϕ^{-1}) and for the quantum potential $\mathfrak{Q}$ (expressed separately via a Lagrangian) with the goal of thereby deriving connections between $\mathfrak{Q}$ and ϕ. Also Bohmian aspects of the theory are emphasized. Some of this is also reviewed in [15, 16]. A main conclusion of this is that the quantum potential is a dynamical field and interactions between Λ and $\mathcal{Q}$ represent a connection between large and small scale structures. We note also from [73] that Λ plays an important role in generating mass for scalar fields and the understanding of Λ is considerably deepened and enhanced via work of Klinkhamer and Volovik (cf. [61, 62, 90, 91]) some of which is described briefly below. ∎

REMARK 2.3. Relations between thermodynamics and gravity have been extensively studied following early work of Bekenstein and Hawking plus more recently Jacobson [57], and Padmanabhan [75, 76]. One can derive general Einstein field equations via thermodynamic principles and

gravity itself seems to be characterized via thermodynamics (see [17] for a brief sketch of some of this and there is much more information in works of Padmanabhan et al. The work of Verlinde [89] has triggered another explosion of interest in entropy and gravity and we mention here only a few articles, e.g. [9, 10, 13, 93] (related to the Friedmann equations) and [67, 70, 72, 88] (relations to quantum mechanics). ∎

We would like to mention here that entropy in quantum mechanics is normally connected to momentum fluctuations and the quantum potential gives rise to an entropy functional $\int P\mathfrak{Q}\,dx$ corresponding to Fisher information (here $P = |\psi|^2$ where ψ is a wave function and $\mathfrak{Q}$ represents a 3-dimensional quantum potential). Generally $\mathfrak{Q}$ can be described e.g. via an osmotic velocity or a thermalization of this (cf. [15, 16, 17, 27, 34, 39, 45, 49, 50, 51] and Section 3). There is also a gravitational version of this related to the Wheeler-deWitt (WDW) framework in the form

$$\int \mathcal{D}h\, P\mathfrak{Q} = \int \mathcal{D}h \frac{\delta P^{1/2}}{\delta h_{ij}} G_{ijk\ell} \frac{\delta P^{1/2}}{\delta h_{k\ell}} \tag{2.12}$$

(cf. [16, 17, 21, 34, 51]) and see also [40, 92] for WDW). We note also the Perelman entropy functional

$$\mathfrak{F} = \int_M (R + |\nabla f|^2) e^{-f} dV \tag{2.13}$$

and corresponding Ricci flows are related to a so called Nash entropy (**2I**) $S = \int u\, log(u)\, dV$ where $u = exp(-f)$ and various aspects of quantum mechanics related to the Schrödinger equation and Weyl geometry arise (cf. [17, 18, 19, 43, 53, 59]).

3. SOME BACKGROUND

First it is interesting and perhaps enlightening to have some 3-D background here on the quantum potential. Thus, referring to [15, 16, 17, 22, 23, 26, 34, 35, 36, 37, 39, 45, 47, 48, 49, 50, 51, 71, 72, 82, 83, 88] we recall some results involving the QP relative to a 3-D SE. Various notations are used and we deal with a Schrödinger equation (SE) (**3A**) $i\hbar\psi_t = \Delta\psi + V\psi$ with $\psi = Rexp(iS/\hbar)$ and

$$\partial_t S + \frac{1}{2m}|\nabla S|^2 + V + \mathfrak{Q} = 0;\ \partial_t(R^2) + \frac{1}{m}\nabla(R^2\nabla S) = 0 \tag{3.1}$$

where the quantum potential (QP) is defined via

$$\mathfrak{Q} = -\frac{\hbar^2}{2m}\frac{\Delta R}{R} = -\frac{\hbar^2}{4m}\left[\frac{1}{2}\left(\frac{\nabla P}{P}\right)^2 - \frac{\Delta P}{P}\right] \tag{3.2}$$

(note this is different from $\mathfrak{Q}$ in e.g. (**2F**)). Here $|\psi|^2 = R^2 = P$ and $\rho = mP$. Setting $D = (\hbar/2m)$ one can write (**3B**) $\mathbf{u} = -D(\nabla P/P)$ (osmotic velocity) which is expressed in terms of a "canonical" momentum fluctuation $\delta p/m = \mathbf{u}$ (cf. [15, 16, 17] for references and more details). Now setting $\mathbf{k}_u = -(1/2)\mathbf{u}$ one can write (**3C**) $\mathfrak{Q} = (\hbar^2/2m)(|\mathbf{k}_u|^2 - \nabla\cdot\mathbf{k}_u) = (m/2)\mathbf{u}^2 - (\hbar/2)(\nabla\cdot\mathbf{u})$. There is also a thermalization of the QP based on [45] (cf. also [17, 22]) whereby one refers to a basic particle energy $E = \hbar\omega$ (where $kT = \hbar\omega = 1/\alpha$ with $k \sim k_B$ the Boltzman constant). One then defines a thermal function $\mathcal{Q}$ where $\Delta\mathcal{Q} = \mathcal{Q}(t) - \mathcal{Q}(0) < 0$ determines a heat dissipation with an osmotic velocity (**3D**) $\mathbf{u} = -D(\nabla P/P) = (1/2\omega m)\nabla\mathcal{Q}$. Hence and from (3.2), (**3B**), and (**3D**) there results

$$\int \mathfrak{Q}Pd^3x = \frac{\hbar^2}{8m}\int\left(\frac{\nabla P}{P}\right)^2 Pd^3x = \frac{m}{2}\int|\mathbf{u}|^2Pd^3x = \tag{3.3}$$

$$= \frac{1}{8\omega^2 m}\int|\nabla\mathcal{Q}|^2Pd^3x = \tilde{c}(t)\int(\nabla\mathcal{Q})^2 exp^{-\alpha\mathcal{Q}}d^3x$$

(the last equation via $log(P) = -\alpha\mathcal{Q} + c(t) \Rightarrow P = \hat{c}(t)exp(-\alpha\mathcal{Q})$ where $\alpha = 1/\hbar\omega$ - cf. [17, 22, 45, 47, 48] for more on this). One should also note that (**3E**) $\int P\mathfrak{Q}d^3x = (\hbar^2/8m)\int[(|\nabla P|^2/P]d^3x = (\hbar^2/8m)FI$ where FI denotes Fisher information.

Now we note that an idea of osmotic pressure arises in the Klinkhamer-Volovik theory of superfluids (cf. [61, 62, 90, 91]) from which we extract here (see especially gr-qc 0711.3170, 0806.2805, 0907.4887, 0811.4347, and 1102.3152; hep-th 0907.4887 and 1101.1281; cond-mat 1004.0597; physics 0909.1044 and cf. also [30, 52, 79, 80]). This is a relativistic theory and the relativistic quantum vacuum is considered as a self sustained medium described by a variable $\mathfrak{q}$ which is a conserved quantity analogous to a particle density n in condensed matter theory but $\mathfrak{q}$ is the relativistic invariant quantity (see also Section 4). There is a Gibbs-Duhem relation $\epsilon_{vac}(\mathfrak{q}) - \mu\mathfrak{q} = -P_{vac}$ where ϵ_{vac} is the vacuum energy density and μ is the vacuum chemical potential (thermodynamically conjugate to $\mathfrak{q}$). Dynamical equations for $\mathfrak{q}$ show that $\Lambda = \epsilon_{vac} - \mu\mathfrak{q} = -P_{vac}$ so the vacuum variable is automatically self-tuned to nullify in equilibrium any contribution from different quantum fields. In [91] (0909.1044) one considers a dilute solution of ^3He quasiparticles in a superfluid vacuum ^4He at $T = 0$. The negative contribution of the vacuum to the osmotic pressure of ^3He is given by the same equation

$$P_{os} = P_{mat} + P_{vac} = P_{mat} - \frac{1}{2}\chi_{vac}\left[\mathfrak{q}\frac{\partial\epsilon_{mat}(n,\mathfrak{q})}{\partial\mathfrak{q}}\right]^2 \tag{3.4}$$

where χ_{vac} is the vacuum compressibility

$$\chi_{vac}^{-1} = \left[\mathfrak{q}^2 \frac{d^2\epsilon_{vac}(\mathfrak{q})}{d\mathfrak{q}^2}\right]_{\mathfrak{q}=\mathfrak{q}_0} \geq 0 \tag{3.5}$$

(where now χ_{vac} corresponds to the compressibility of liquid ^4He. This negative contribution to the osmotic pressure has been experimentally verified (according to 0909.1044).

REMARK 3.1 . Note that the formula in (3.3) involving the 3-D integral $\int |\nabla\mathcal{Q}|^2 exp(-\alpha\mathcal{Q})d^3x$ reminds one of the I_2 formula (1.13) from S_4 where there is a 4-D integral involving $exp(-\psi)|\hat{\nabla}\psi|^2$; also the S_4 or I_2 represent "action" terms in the Weyl theory as does $\int P\mathfrak{Q}d^3x$ in the 3-D theory (cf. [15, 16, 17]). However there is no immediate extrapolation here from 3-D to 4-D or vice-versa. In fact we know from remarks before (2.11) that $\psi \sim \mathfrak{Q}$ in the 4-D theory where $\mathfrak{Q}$ is given by (**2F**) and in the 3-D theory there is a formula (cf. [17, 45])

$$\mathfrak{Q} = -\frac{\hbar^2}{4k_BTm}\left[\nabla^2\mathcal{Q} - \frac{1}{D}\partial_t\mathcal{Q}\right] \tag{3.6}$$

We refer to Section 4 for some remarks on the KG equation and recall that there is related information via the Wheeler-deWitt (WDW) or Klein-Gordon (KG) equations in e.g. [6, 17, 21, 32, 34, 46, 51]. ∎

REMARK 3.2. The role of information theory and entropy in QM has been clarified by e.g. Caticha [26] (see also in this connection [7, 71, 72, 88]). Osmotic velocity plays an important role in the diffusion theory of course and we mention also some recent work of Munkhammar [70] related to [89]. One can argue that sometimes QM is signaled simply by the entrance of $\hbar$ into the theory. For example in the holographic force scenario of Verlinde [89] it seems to arise via the Unruh temperature relation to acceleration (**3F**) $k_BT = (1/2\pi)(\hbar a/c)$. Padmanabhan et al have written deeply and extensively about matters of entropy, entanglement, equipartition, horizons, and gravity as an emergent phenomenon (cf. [75, 76]). Entanglement entropy is also studed by J.W. Lee et al in [60, 63, 64] and one theme involves entropy and information erasure. An interesting Machian point of view is developed in [41, 42] and in [97] there is a microscopic approach to entropic gravity via the idea of coherence length. In [45] one defines a thermal term s via $\nabla\mathcal{Q} = 2\omega\nabla(\delta s) = 2\omega m\mathbf{u} = -\omega\hbar(\nabla P/P)$, providing the osmotic velocity, and in [26] this is combined with a relative entropy term $\mathfrak{S}$, based on "hidden" variables, for the "drift" velocity and a current velocity determined via ϕ where $\phi = \mathfrak{S} - log(\rho^{1/2})$ with velocity connection $v^a = b^a + u^a$

(see [26] for details). In [58] it is also shown that in entropic quantum dynamics gravitational potential and accelerating frames are informationally equivalent. ■

4. SUPERFLUIDS

The superfluid universe of Klinkhamer and Volovik (cf. especially [91] (cond-mat 1004.0597) and [62] (0811.4347) is a fascinating theme and we mention a few mathematical points here which were designed to give some structure to the superfluid world. We have a very imperfect and fragmentary idea of the physics here and only look at certain mathematical features which seem related to the material sketched above (and in [15, 16, 17] etc.). As an example of the arena for $\mathfrak{q}$ theory one looks for $\mathfrak{q}^{\mu\nu}$ which in a homogeneous vacum has the form (**4A**) $\mathfrak{q}^{\mu\nu} = \mathfrak{q}g^{\mu\nu}$ or (**4B**) $\mathfrak{q}_{\mu\nu\alpha\beta} = \mathfrak{q}(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu})$ (generally here $\hbar = c = k_B = 1$). Another example is (**4C**) $\mathfrak{q}^{\mu\nu\alpha\beta} = \mathfrak{q}\epsilon^{\mu\nu\alpha\beta}$ where $\epsilon^{\mu\nu\alpha\beta}$ is the fully antisymmetric Levi-Civita tensor (cf. [14]). More specifically one can consider a chiral condensate of gauge fields (e.g. a gluonic condensate in QCD). Assume e.g. that $F_{\alpha\beta}$ represents a color magnetic field with (**4D**) $< F_{\alpha\beta} >= 0$ and $< F_{\alpha\beta}F_{\mu\nu} >= (\mathfrak{q}/24)\sqrt{-g}\epsilon_{\alpha\beta\mu\nu}$ where $\mathfrak{q}$ is the anomaly driven topological condensate

$$\mathfrak{q} =< \tilde{F}^{\mu\nu}F_{\mu\nu} >= \frac{1}{\sqrt{-g}}\epsilon^{\alpha\beta\mu\nu} < F_{\alpha\beta}F_{\mu\nu} > \tag{4.1}$$

Then one chooses a vacuum action (**4E**) $S_{\mathfrak{q}} = \int d^4x\sqrt{-g}\epsilon(\mathfrak{q})$ leading to a stress-energy tensor

$$T^{\mathfrak{q}}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathfrak{q}}}{\delta g^{\mu\nu}} = \epsilon(\mathfrak{q})g_{\mu\nu} - 2\frac{\partial\epsilon}{\partial\mathfrak{q}}\frac{\partial\mathfrak{q}}{\partial g^{\mu\nu}} \Rightarrow \frac{\partial q}{\partial g^{\mu\nu}} = \frac{1}{2}qg_{\mu\nu} \Rightarrow \tag{4.2}$$
$$\Rightarrow T^{\mathfrak{q}}_{\mu\nu} = g_{\mu\nu}\rho_{vac}(\mathfrak{q});\ \ \rho_{vac}(\mathfrak{q}) = \epsilon(\mathfrak{q}) - \mathfrak{q}\frac{\partial\epsilon}{\partial\mathfrak{q}}$$

In terms of the Einstein equations one has then (**4F**) $T^{\mathfrak{q}}_{\mu\nu} = \Lambda g_{\mu\nu};\ \Lambda = \rho_{vac}(\mathfrak{q}) = \epsilon(\mathfrak{q}) - \mathfrak{q}\frac{\partial\epsilon}{\partial\mathfrak{q}}$. Other possibilities are indicated in [62, 91] but fulfillment of (4.2) and (**4F**) is essential. Recall also from Section 3 that $\epsilon_{vac} - \mu\mathfrak{q} = -P_{vac}$ where μ is the chemical potential and one can argue that $-P_{vac} \sim \Lambda \sim \rho_{vac}$.

We recall now, from the discussion in Section 3, that given an osmotic velocity $\mathbf{u}$ in some region $\Omega \in \mathbf{R}^3$ there will be a local osmotic pressure

$$P_{loc} = \frac{1}{2}\rho\mathbf{u}^2 = \frac{m}{2}PD^2\frac{|\nabla P|^2}{P^2} = \frac{\hbar^2}{8m}\frac{|\nabla P|^2}{P} \tag{4.3}$$

so that (**4G**) $P_{osm} = \int_\Omega P_u d^3x = \int \mathfrak{Q}Pd^3x = (\hbar^2/8m)FI$. In the superfluid universe mentioned above (with vacuum ^4He) one might to suggest that

this could correspond to a vacuum osmotic pressure induced by a quantum generated "vacuum" based on Sections 1-2. In order to develop this one should examine the osmotic pressure idea in a 4-D context, in view of the expression (**2F**) for $\mathfrak{Q}$.

We sketch now from [46] (quant-ph 0201035) where the Nelsonian perspective is extended to a deBroglie-Bohm (BB) in the context of describing a KG framework (cf. also [65]). Perhaps somewhat prophetically it is mentioned in [46] (quant-ph 0201035) that the quantum vacuum has to play a major role in the evolution of the universe and one must take Λ seriously (which directive is being followed today, especially in the theory of superfluids). The introduction of an aether is appropriate in a relativistic (BB) theory and enters into the KG construction. We extract now from [46] (quant-ph 0201035) a few equations regarding osmotic velocity in hopes of implementing in some manner the superfluid theory. Thus, following [46], we write (**4H**) $\psi(x,t) = R(x,t)exp(-iS/\hbar)$; $[\Box + (m^2c^2/\hbar^2)]\psi = 0$ leading to $(\Box = (1/c^2)\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2)$

$$\frac{\Box R}{R} + \frac{1}{\hbar}\frac{\partial_\mu R}{R}\partial^\mu S + \frac{i}{\hbar}\frac{\partial^\mu R}{R}\partial_\mu S - \frac{1}{\hbar^2}\partial_\mu S\partial^\mu S + \frac{i}{\hbar}\Box S + \frac{m^2c^2}{\hbar^2} = 0 \tag{4.4}$$

from which follows (for $P = R^2$ and $J_\mu = P\partial_\mu S$)

$$2\frac{\partial^\mu R}{R}\partial_\mu S + \Box S = 0;\ \partial^\mu J_\mu = \partial^\mu P\partial_\mu S + P\Box S = 0 \tag{4.5}$$

along with (**4I**) $\partial_\mu S\partial^\mu S = M^2c^2 = m^2c^2 + \hbar^2(\Box R/R)$. Now to include vacuum energy one writes (**4J**) $S^0 = E_{vac}t$ along with (**4K**) $\Psi(x,t) = R(x,t)exp[-i(S+S^0)/\hbar]$ for a solution of the KG equation. Then instead of (4.5) one involves a "hidden" probability current $J^0_\mu = P\partial_\mu S^0$ to get a conservation law for the total probability current (**4L**) $\partial^\mu(J_\mu + J^0_\mu) = 0$ where (**4M**) $\partial^\mu J_\mu = \partial^\mu P\partial_\mu S + P\Box S = -\partial^\mu P\partial_\mu S^0 - P\Box S^0 = -\partial^\mu J^0_\mu$. Further S^0 also modifies (**4I**) leading to a variable total mass

$$M^2c^2 \to \left[mc + \frac{E_{vac}}{c}\right]^2 + \hbar^2\frac{\Box R}{R} \tag{4.6}$$

Thus a ("hidden") diffusion current J^0_μ is introduced, due to the assumed stochastic aether dynamics and a corresponding all-pervading vacuum energy E_{vac}. One expects that timelike trajectories will ensue, i.e. the zero component $J^0_{vac} = P\partial_t(S+S^0) \geq 0$.

Now in order to relate this causal KG theory with Nelson's stochastic theory one defines (**4N**) $S = Et - \mathbf{px} = \hbar p_\mu x^\mu$ such that $\mathbf{v}^\mu = \partial^\mu S/M$. Also an external potential V is assumed (which could include the vacuum energy E_{vac}) with M given by (**4I**) as ($\bullet$) $M = \{[m + (V/c^2)]^2 +$

$(\hbar^2/c^2)(\Box R/R)]\}^{1/2}$. Thus the two real valued equations corresponding to the KG equation involves (**4O**) $\partial^\mu J_\mu = \partial^\mu P\partial_\mu S + P\Box S = 0$ and a Hamilton- Jacobi-Bohm equation

$$\partial_\mu S\partial^\mu S = M^2\mathbf{v}_\mu\mathbf{v}^\mu = M^2c^2 = \left[mc + \frac{V}{c}\right]^2 + \mathfrak{Q} \tag{4.7}$$

where (**4P**) $\mathfrak{Q} = \hbar^2(\Box R/R)$. This leads to an "osmotic velocity" (**4Q**) $\mathbf{u}^\mu = D[\partial^\mu P/P]$ with $D = \hbar/2m$ (cf. also [46] - book). One can then rewrite (**4O**) and (4.7) respectively as

$$\partial_\mu\mathbf{v}^\mu = -\frac{1}{D}\mathbf{u}_\mu\mathbf{v}^\mu - \frac{\partial_\mu M}{M}\mathbf{v}^\mu;\ M^2c^2 = \left[mc + \frac{V}{c}\right]^2 + \mathfrak{Q} \tag{4.8}$$

where (**4R**) $\mathfrak{Q} = m^2c^2 + m\hbar\partial_\mu\mathbf{u}^\mu$ showing that the dynamics is essentially governed by the 4-gradient of the osmotic velocity. Also there is then a local osmotic pressure based on $(1/2)M|\mathbf{u}|^2$ with M as in ($\bullet$). In the sense of [65] (but now in explicit dependence on the osmotic velocity) equations (4.8) are the two basic equations of relativistic stochastic mechanics (cf. [46] for more on this). Note that one can also rewrite (4.8-A) as

$$\partial_\mu\mathbf{v}^\mu = -\frac{1}{D}(\mathbf{u}_\mu + \mathbf{u}_\mu^M)\mathbf{v}^\mu;\ \mathbf{u}_\mu^M = D\frac{\partial_\mu M}{M} \tag{4.9}$$

We refer to [46] for more equations and discussion and anticipate possible interesting connections of the analysis here to superfluid theory.

The obvious question now is to determine a possible relation between M and $\mathfrak{M}$ where $\mathfrak{M}$ is the quantum mass of Section 1 and M is the relativistic mass of [46]. We recall that the Weyl geometry involved $\hat{g}_{ab} = \Omega^2(x)g_{ab}$ with(**4S**) $\mathfrak{M}^2 = m^2exp(\mathfrak{Q}$ and (**4T**) $\mathfrak{Q} = (\hbar^2/m^2)(\Box|\Psi|/|\Psi|)$. On the other hand there is no Weyl geometry explicit in Section 4 and one has from ($\bullet$), (4.7), (4.8)

$$M^2c^2 = \left[mc + \frac{V}{c}\right]^2 + \mathfrak{Q};\ \mathfrak{Q} = \hbar^2\frac{\Box R}{R} \tag{4.10}$$

Now we recall that when working directly with a KG equation it is natural to use an approximation (**4U**) $exp(\mathfrak{Q} \sim 1 + \mathfrak{Q}$ (cf. [15, 16, 87]) and (**4S**) becomes then (**4V**) $\mathfrak{M}^2 \sim m^2(1 + \mathfrak{Q})$. Setting $c = 1$ in (4.10) then one can compare

$$M^2 = (m + V)^2 + \hbar^2\frac{\Box R}{R}\ \ and\ \ \mathfrak{M}^2 = m^2\left[1 + \frac{\hbar^2}{m^2}\frac{\Box R}{R}\right] \tag{4.11}$$

Evidently $\mathfrak{M}^2 \sim M^2$ with $m \to m + V$ where V contains E_{vac}. If we take $V = E_{vac}$ then M only differs from $\mathfrak{M}$ by an E_{vac} term, i.e. for $\tilde{\mathfrak{Q}} = \hbar^2\Box R/R$

$$\sqrt{M^2 - \tilde{\mathfrak{Q}}} - \sqrt{\mathfrak{M}^2 - \tilde{\mathfrak{Q}}} = E_{vac} \tag{4.12}$$

In a certain sense then the effect of relativity here is to display the influence of a quantum vacuum.

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